Equal Compositions of Rational Functions

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The Problems

A rational function is a ratio of two polynomials.

Problem 1

Find all rational functions $a, c \in \mathbb{Q}(X)$ such that $a(Y) = c(Z)$ has infinitely many solutions for $Y, Z \in \mathbb{Q}$.

One source of solutions to Problem 1 comes from the following problem when the functions have rational coefficients:

Problem 2

Find all rational functions $a, b, c, d \in \mathbb{C}(X)$ such that

$$a(b(X)) = c(d(X)).$$

Some examples:

- $X^m \circ X^n = X^n \circ X^m = X^{mn}$

SOME EXAMPLES:

- $X^m \circ X^n = X^n \circ X^m = X^{mn}$
- For an arbitrary rational function $h(X)$,

$$X^2 \circ Xh(X^2) = Xh(X)^2 \circ X^2 = X^2 h(X^2)^2.$$

Result

Theorem

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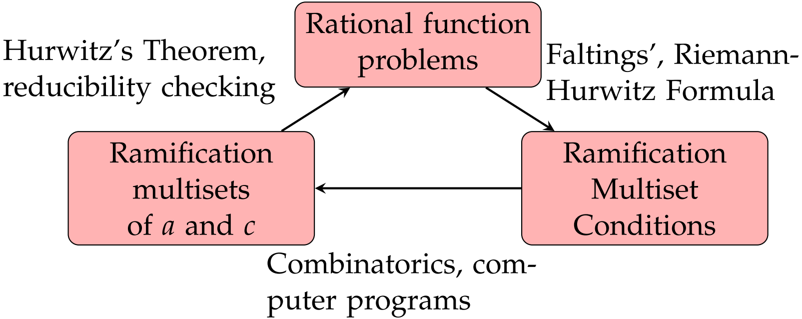
- $\deg a, \deg c \leq 250$
- *at least one of a and c are "nice" functions (e.g. X^m, Chebyshev, functions coming from elliptic curves)*
- *Up to change in variables,*

$$a = X^i(X-1)^j, c = rX^i(X-1)^j.$$

Outline of our strategy



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- Example: $f(X) = X^3 + X^4 = X^3(X+1)$ has $E_f(0) = [3, 1]$.

Multiset problem

The multiset problem

If the numerator of $a(X) - c(Y)$ is irreducible,

N.1. $\sum_{i \in A_k} i = m$ and $\sum_{i \in C_k} i = n$ for each k (m and n are the degrees of a and c and A_k and C_k are ramification multisets of a and c).

N.2. $\sum_{k=1}^{r} (m - |A_k|) = 2m - 2$ and $\sum_{k=1}^{r} (n - |C_k|) = 2n - 2$.

N.3. $\sum_{k=1}^{r} \sum_{i \in A_k} \sum_{j \in C_k} (i - \gcd(i,j)) \in \{2m-2, 2m\}$.

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- Globally: We find all the possibilities for $\{k_i\}$.
- For each possibility of $\{k_i\}$, we solve for $\{A_i\}$.

Results

Proposition

If rational functions a and c are solutions to the multiset problem, then at least one of a and c satisfies

$$\sum_{k=1}^{r}\left(1-\frac{1}{\operatorname{lcm}(F_k)}\right)\leq 2$$

where $\{F_k\}$ is the list of all ramification multisets of that function.

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9. (u) where u is any integer

SOLVING FOR THE A_i

1. $A_1 \cup A_2 \cup A_3 \cup A_4 = [1^4, 2^{2m-2}]$.

8. $A_1 = A_2 = [m]$.

SOLVING FOR THE C_i

For example, suppose that $A_1 = A_2 = [m]$. This corresponds to $a(X) = X^m$.

1. $c(X) = h(X)^m X^k$ for k relatively prime to m,
2. $m = 6$ and $c(X) = h(X)^6 X^3 (X-1)^{\pm 2}$,
3. $m = 4$ and $c(X) = h(X)^4 X^2 (X-1)^{\pm 1}$,
4. $m = 3$ and $c(X) = h(X)^3 X^{\pm 1} (X-1)^{\pm 1}$ (with the $\pm$ independent),
5. $m = 2$ and $c(X)) = h(X)^2 X (X-1)(X-X_0)$ (with $0 \neq x_0 \neq 1$,

where $h(X)$ is any rational function.

Back to the original problems

- checking that functions a and c exist.

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- determining whether $a(X) - c(Y)$ is irreducible

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Hurwitz's Theorem

A finite collection of k multisets A_i of sum n with corresponds to a rational function if and only if both of the following are true:

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A finite collection of k multisets A_i of sum n with corresponds to a rational function if and only if both of the following are true:

- $\sum_{i \leq k}(n - |A_i|) = 2n - 2$.
- There exist permutations $g_1, \ldots, g_k \in S_n$ such that g_i has cycle structure A_i and the product of the permutations is the identity. Furthermore, the group generated by $g_1, \ldots g_k$ must be transitive.

Testing for irreducibility

Extra Condition

For all $i, j \leq r$, $A_i \cup A_j \cup C_i \cup C_j$ has greatest common divisor equal to one.

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Theorem (Reducibility test)

If $\sum_{k=1}^{r} \sum_{i \in A_k} \sum_{j \in C_k} (i - \gcd(i,j)) < 2m - 2$, *any rationals* $a(X)$ *with multisets* A_k *and* $c(Y)$ *with multisets* C_k *will have* $a(X) - c(Y)$ *reducible.*

This is similar to one of our previous conditions, so we usually keep c the same and vary a to show that c is decomposable so that $a(X) - c(Y)$ is reducible.

FUTURE RESEARCH

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- Finish finding a and c for the case in which a's multisets have small lcm.
- Continue to lower the bounds for 250 and 10 above.
- The case in which $a(X) - c(Y)$ is not irreducible.

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